THE UNIVERSITY OF ALBERTA

THE LIMITING DISTRIBUTION OF THE MEASURE OF A RANDOM SET

by

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ABSTRACT

W.L: Stevens initiated research in the general area of random sets by considering arcs of length D $(0 < D < 1)$ at random points on the circumference of a circle of length unity and derived the probability that the circumference is completely covered. D.F. Votaw Jr. considered a somewhat similar but non-circular problem and derived the probability function of the measure of a random linear set. D.D. Beck, using the methods of Votaw, derived the probability function of the measure of a random circular set. The results of these authors are reviewed in the first chapter.

The second chapter is devoted to finding the limiting distribution and the non-central moments of the measure of a random circular set for the case $D = 1/(n+1)$ where n is the number of arcs. The last section of this chapter contains discussions of plots and tables of the probability function of the measure of a random circular set for various values of n and D.

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TABLE OF CONTENTS

CHAPTER I

INTRODUCTION

Consider $n+1$ points z_i $(i = 0,1,\ldots,n)$ selected independently and at random from the interval $(0,1)$; the distribution of any z_i being the uniform distribution with distribution function

$$F(z) = z , \qquad 0 \leq z \leq 1 . \tag{1.1}$$

Order the $n+1$ points in ascending order of magnitude as x_i $(i = 0,1,\ldots,n)$. The totality of all cases in which two of the z_i's are equal has zero probability and such cases can be excluded without affecting the problem. Associate with each x_i an interval I, where

$$I = \begin{cases} (x_i, x_i+D) & \text{if} \quad x_i+D \leq 1 , \\ (x_i,1) \cup (0, x_i+D-1) & \text{if} \quad x_i+D > 1 ; \end{cases} \tag{1.2}$$

where $0 < D < 1$ and $i = 0,1,\ldots,n.$

Let X denote the random set which is the point set sum of the $(n+1)$ intervals $\{I\}$ and let $\mu(X)$ be its measure. The range of $\mu(X)$ is $D \leq \mu(X) \leq m$, where m denotes the minimum of 1 and $(n+1)D$.

D.D. Beck in his M. Sc. thesis (1968) derived the probability function of $\mu(X)$ using the methods of D.F. Votaw Jr. (1946). The probability function, $f_n(x)$, of the random variable $\mu(X)$ is given by

$$(1.3)\qquad f_n(x) = n! \sum_{j=1}^{q} \sum_{r=0}^{q-j} \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j} \binom{n-j+1}{r} \cdot (1-x)^{j-1} [x-D(j+r)]^{n-j} ,$$

where

$qD \leq x < (q+1)D, \quad q = 1,2,\ldots,M$;

$M = \text{minimum } (n, [1/D])$,

$[a] =$ greatest integer less than or equal to a ,

and $x < m$.

When $m = (n+1)D$, there is a finite probability that the random set X will consist of $n+1$ non-overlapping intervals $\{I\}$ and hence the distribution function, $F_n(x)$, of the variable $\mu(X)$ has a discontinuous jump or saltus point in the case when $m = (n+1)D$ at $x = m$. At such a saltus point, the probability function $f_n(x)$, of $\mu(X)$, is not defined but at all other points it defines the probability density. The expected value of $\mu(X)$ is given by

$$(1.4)\qquad E(\mu(X)) = 1 - (1-D)^{n+1}$$

and the variance is given by

$$(1.5)\qquad \operatorname{Var}(\mu(X)) = \begin{cases} \frac{1}{n+2}\{2(1-D)^{n+2}+n(1-2D)^{n+2}\} - (1-D)^{2n+2}, & 0 < D \leq \frac{1}{2} ; \\ \frac{2}{n+2}(1-D)^{n+2}-(1-D)^{2n+2}, & \frac{1}{2} < D < 1 . \end{cases}$$

For $D = 1/(n+1)$, we show in the second section of

Chapter II that the k^{th} non-central moment, m_k, of the random variable $[1-\mu(X)]$ is given by

$$(1.6) \qquad m_k = \frac{1}{\binom{n+k}{k}} \sum_{i=1}^{k} \binom{n+1}{k} \binom{k-1}{i-1} \left(1 - \frac{i}{n+1}\right)^{n+k} , \quad k = 1,2,\ldots,n+1 .$$

In the last section of Chapter II we plot and tabulate the function $f_n(x)$ for various values of n and D. In fact, we attempted to find the limiting distribution of the standardized variable $\mu(X)$ in the case $D = 1/(n+1)$ as n tends to infinity, which amounts to increasing the number of intervals and at the same time contracting them. Although we were unable to determine it exactly, our results suggest that the limiting distribution of the variable obtained by standardizing $\mu(X)$ is standard normal. We give the method of analysis we were attempting and indicate where difficulties arose, in the second chapter. It is unfortunate that we have been unable to complete the analysis of the problem, but the partial analysis gives strong evidence to suggest that the limiting distribution is normal.

Research, which one might classify under the general heading of the measure of a random set, was initiated by a paper in 1939 by W.L. Stevens. Beck, in his thesis (1968), gives a good account of the various papers published in this area since then. Consider the following problem: take $n+1$ arcs of length D, $0 < D < 1$, at random points on a circle of unit circumference. Let the location of each arc be represented by the point at its clockwise

end. Choose the end of one arc arbitrarily as the origin and measure distances anticlockwise around the circle. If we consider the point set sum of all the arcs then this representation of the random set is equivalent to the linear representation described earlier. The question is what is the probability that every point of the circle is included in at least one of the arcs? Or, in other words, what is the probability that the circle is completely covered? Stevens (1939) solved this problem and proved that the required probability is given by

$$(1.7) \qquad 1-\binom{n+1}{1}(1-D)^n+\binom{n+1}{2}(1-2D)^n + \ldots + (-1)^k\binom{n+1}{k}(1-kD)^n ,$$

where $k = [1/D]$.

He also calculated the frequency distribution of the number of gaps. There is said to be a gap after the r^{th} arc if for a distance greater than D beyond the first point of this arc, there is no other arc.

In the remainder of this chapter we give the results obtained by D.F. Votaw Jr. (1946). Using the methods given in this paper, Beck (1968) derived (1.3).

Votaw considered the following non-circular representation: take a random sample z_i $(i = 1,2,\ldots,n)$ of n values of a one dimensional random variable Z having distribution function $F(z)$. Arrange the values in increasing order of magnitude as x_i, $i = 1,2,\ldots,n$ and consider the set X consisting of the point set sum of the following intervals:

(1.8) $\quad (x_i - \frac{D}{2}, x_i + \frac{D}{2}) \ , \quad i = 1,2,\ldots,n, \quad$ and $\quad 0 < D < 1 \ .$

This representation of the random set X is not equivalent to the linear representation described earlier, which was equivalent to the circular representation. The difference arises when we consider the intervals at the end-points. Let $\mu(X)$ be the measure of X and define a random variable y as $y = \mu(X)-D$. Then the range of y is $0 \le y \le m_1$, where m_1 denotes the minimum of 1 and $(n-1)D$. Votaw showed that if $F(z)$ is given by (1.1) then the probability function $f_n(y)$ of y is given by

(1.9)
$$f_n(y) = n \sum_{j=0}^{q} \sum_{r=0}^{q-j} (-1)^r \binom{n-1}{j} \binom{n-1}{j+1} \binom{n-j-1}{r} \cdot (1-y)^{j+1} \{y-D(j+r)\}^{n-j-2} \ ,$$

where

$$qD \le y < (q+1)D \ ,$$

$$q = 0,1,\ldots,M_1 \ ;$$

$$M_1 = \text{minimum } (n-2,[1/D]), \quad \text{and} \quad y < m_1 \ .$$

He also showed that the expected value of y is given by

(1.10)
$$E(y) = \frac{(n-1)}{(n+1)} [1 - (1-D)^{n+1}] \ .$$

Rewriting (1.9) we get

$$(1.11) \qquad f_n(y) = n! \sum_{j=0}^{q} \sum_{r=0}^{q-j} \frac{(-1)^r}{(n-1-j)!j!} \binom{n-1}{j+1} \binom{n-j-1}{r} \cdot (1-y)^{j+1} \{y - D(j+r)\}^{n-j-2} \quad ,$$

with the same conditions as in (1.9). Comparing (1.3) and (1.11), we see a lot of resemblance and the differences may be attributed to the following reasons:

(i) the two representations of the random set are not equivalent and hence we have two different random variables,

(ii) Beck considered $(n+1)$ intervals instead of the n considered by Votaw, and

(iii) Beck derived the distribution of the random variable $\mu(X)$, whereas Votaw derived the distribution of $\mu(X)-D$.

CHAPTER II

MAIN RESULTS

In this chapter, we present our attempt to find the limiting distribution of the random variable $\mu(X)$. The case $D = 1/(n+1)$ is of particular interest because this amounts to reducing the size of each interval I and at the same time increasing the total number of such intervals in such a fashion that it is just possible for the circle to be completely covered by the arcs.

2.1 Preliminary

When $D = 1/(n+1)$, the expressions (1.3), (1.4) and (1.5) for the probability function $f_n(x)$, the expected value and the variance of $\mu(X)$ respectively reduce to

$$\text{(2.1.1)} \qquad f_n(x) = n! \sum_{j=1}^{q} \sum_{r=0}^{q-j} \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j} \binom{n-j+1}{r} \cdot (1-x)^{j-1} \left[x - \frac{j+r}{n+1}\right]^{n-j} ,$$

where

$$\frac{q}{n+1} \leq x < \frac{q+1}{n+1} , \qquad q = 1,2,\ldots,n \qquad \text{and} \qquad x < 1 ;$$

$$\text{(2.1.2)} \qquad E(\mu(X)) = \mu_n = 1 - \left(1 - \frac{1}{n+1}\right)^{n+1}$$

and

$$\text{(2.1.3)} \qquad \mathrm{Var}\,(\mu(X)) = \sigma_n^2 = \frac{1}{n+2} \left\{2\left(1 - \frac{1}{n+1}\right)^{n+2} + n\left(1 - \frac{2}{n+1}\right)^{n+2}\right\} - \left(1 - \frac{1}{n+1}\right)^{2n+2} ,$$

as $0 < \frac{1}{n+1} \leq \frac{1}{2}$, for all $n \geq 1$.

Consider the limiting behaviour of the expected value and the variance of the random variable $\mu(X)$. From (2.1.2),

$$(2.1.4) \qquad \lim_{n \to \infty} \mu_n = \lim_{n \to \infty} \{1 - (1 - \frac{1}{n+1})^{n+1}\} = 1 - e^{-1} ,$$

that is, the expected value tends to a constant as n goes to infinity. However, from (2.1.3),

$$(2.1.5) \qquad \lim_{n \to \infty} \sigma_n^2 = \lim_{n \to \infty} \{\frac{1}{n+2} [2(1 - \frac{1}{n+1})^{n+2} + n(1 - \frac{2}{n+1})^{n+2}] - (1 - \frac{1}{n+1})^{2n+2}\}$$

$$= \lim_{n \to \infty} \frac{1}{n+2} \{2[(1 - \frac{1}{n+1})^{n+2} - (1 - \frac{1}{n+1})^{2n+2}] + n[(1 - \frac{2}{n+1})^{n+2} - (1 - \frac{1}{n+1})^{2n+2}]\}$$

$$= e^{-2} - (e^{-1})^2 = 0 .$$

Since the limiting variance is zero, the limiting distribution of $\mu(X)$ appears to be concentrated on the limiting mean value $(1-e^{-1})$. Let us try to find the rate at which σ_n^2 tends to zero.

Lemma 2.1:

$$\lim_{n \to \infty} (n+2)\sigma_n^2 = 2e^{-1} - 5e^{-2} .$$

Proof: From (2.1.5),

$$\sigma_n^2 = \frac{2}{n+2}\left\{(1-\frac{1}{n+1})^{n+2} - (1-\frac{1}{n+1})^{2n+2}\right\}$$

$$+ \frac{n}{n+2}\left\{(1-\frac{2}{n+1})^{n+2} - (1-\frac{1}{n+1})^{2n+2}\right\} .$$

Now

$$(1-\frac{2}{n+1})^{n+2} - (1-\frac{1}{n+1})^{2n+2}$$

$$= (1-\frac{2}{n+1})^{n+2} - \left(1-\frac{2}{n+1}+\frac{1}{(n+1)^2}\right)^{n+1}$$

$$= (1-\frac{2}{n+1})^{n+1}\left\{(1-\frac{2}{n+1}) - \left(1+\frac{1}{(n+1)(n-1)}\right)^{n+1}\right\}$$

$$= (1-\frac{2}{n+1})^{n+1}\left\{1-\frac{2}{n+1} - 1 - \frac{n+1}{(n+1)(n-1)}\right.$$

$$\left. - \frac{(n+1)\,n}{2(n+1)^2(n-1)^2} - \frac{(n+1)n(n-1)}{6(n+1)^3(n-1)^3} \quad \ldots\right\} .$$

Thus

$$\lim_{n\to\infty} (n+2)\sigma_n^2 = \lim_{n\to\infty} 2\left\{(1-\frac{1}{n+1})^{n+2} - (1-\frac{1}{n+1})^{2n+2}\right\}$$

$$+ \lim_{n\to\infty}\left\{-\frac{2n}{n+1} - \frac{n(n+1)}{(n+1)(n-1)} - 0(\frac{1}{n})\right\}$$

$$\cdot\ (1-\frac{2}{n+1})^{n+1}$$

$$= 2(e^{-1} - e^{-2}) + (-3)e^{-2} = 2e^{-1} - 5e^{-2} .$$

2.2 Limiting Distribution

To find the limiting distribution of a sequence of absolutely continuous distributions it suffices to consider the

limit of their moment generating functions $\{M_n(t)\}$, where

$$M_n(t) = E(e^{t\mu(X)}) = \int_R e^{tx} f_n(x)\, dx \quad , \tag{2.2.1}$$

R being the range of $\mu(X)$. Thus

$$M_n(t) = \int_{1/(n+1)}^{1} e^{tx} f_n(x)\, dx$$

$$= \sum_{\ell=1}^{n} \int_{\ell/(n+1)}^{(\ell+1)/(n+1)} e^{tx} f_n(x)\, dx \quad .$$

Substituting the expression for $f_n(x)$, we get

$$M_n(t) = \sum_{\ell=1}^{n} \int_{\ell/(n+1)}^{(\ell+1)/(n+1)} n! \sum_{j=1}^{\ell} \sum_{r=0}^{\ell-j} \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j} \binom{n-j+1}{r}$$

$$\cdot (1-x)^{j-1} \left(x - \frac{j+r}{n+1}\right)^{n-j} e^{tx}\, dx$$

$$= n! \sum_{\ell=1}^{n} \sum_{j=1}^{\ell} \sum_{r=0}^{\ell-j} \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j} \binom{n-j+1}{r}$$

$$\cdot \int_{\ell/(n+1)}^{(\ell+1)/(n+1)} e^{tx} (1-x)^{j-1} \left(x - \frac{j+r}{n+1}\right)^{n-j} dx \quad .$$

Let $(n+1)x = y$. Then

$$M_n(t) = \frac{n!}{(n+1)} \sum_{\ell=1}^{n} \sum_{j=1}^{\ell} \sum_{r=0}^{\ell-j} \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j} \binom{n-j+1}{r} \tag{2.2.2}$$

$$\cdot \int_{\ell}^{\ell+1} e^{ty/(n+1)} \left(1 - \frac{y}{n+1}\right)^{j-1} \left(\frac{y}{n+1} - \frac{j+r}{n+1}\right)^{n-j} dy$$

$$= \frac{n!}{n+1} \sum_{\ell=1}^{n} \sum_{j=1}^{\ell} \sum_{r=0}^{\ell-j} g_{j,r} \int_{\ell}^{\ell+1} h_{j,r}\, dy ,$$

where

$$g_{j,r} = \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j} \binom{n-j+1}{r}$$

and

$$h_{j,r} = e^{ty/(n+1)} \left(1 - \frac{y}{n+1}\right)^{j-1} \left(\frac{y}{n+1} - \frac{j+r}{n+1}\right)^{n-j} .$$

Consider the triple sum over ℓ, j and r,

$$S = \sum_{\ell=1}^{n} \sum_{j=1}^{\ell} \sum_{r=0}^{\ell-j} g_{j,r} \int_{\ell}^{\ell+1} h_{j,r}\, dy .$$

When $\ell = 1$, we must have $j = 1$ and $r = 0$ and the contribution to S is therefore $g_{1,0} \int_1^2 h_{1,0}\, dy$. When $\ell = 2$, we must have $j = 1$ with $r = 0,1$, or $j = 2$ with $r = 0$; the contributions to S from these two cases are therefore

$$g_{1,0} \int_2^3 h_{1,0}\, dy + g_{1,1} \int_2^3 h_{1,1}\, dy$$

and

$$g_{2,0} \int_2^3 h_{2,0}\, dy , \quad \text{respectively.}$$

Continuing in this manner, it is readily verified that the contributions of the various terms resulting from the permitted combinations of (ℓ,j,r) result in

$$S = g_{1,0}\int_1^{n+1} h_{1,0}\,dy + g_{1,1}\int_2^{n+1} h_{1,1}\,dy + \ldots + g_{1,n-1}\int_n^{n+1} h_{1,n-1}\,dy$$

$$+ g_{2,0}\int_2^{n+1} h_{2,0}\,dy + \ldots + g_{2,n-2}\int_n^{n+1} h_{2,n-2}\,dy$$

$$+ \ldots \quad \ldots \quad \ldots$$

$$+ g_{n,0}\int_n^{n+1} h_{n,0}\,dy$$

$$= \sum_{j=1}^{n}\sum_{r=0}^{n-j} g_{j,r}\int_{j+r}^{n+1} h_{j,r}\,dy \quad .$$

Thus

$$(2.2.3) \qquad M_n(t) = \frac{n!}{n+1}\sum_{j=1}^{n}\sum_{r=0}^{n-j} \frac{(-1)^r}{(n-j)!(j-1)!}\binom{n+1}{j}\binom{n-j+1}{r}$$

$$\cdot \int_{j+r}^{n+1} e^{ty/(n+1)}\left(1-\frac{y}{n+1}\right)^{j-1}\left(\frac{y}{n+1}-\frac{j+r}{n+1}\right)^{n-j} dy \; .$$

Let $z = \frac{y-(j+r)}{n+1-j-r}$, so that $0 \le z \le 1$ and $y = [(n+1)-j-r]z + j+r$, $dy = (n+1-j-r)dz$. Then

$$M_n(t) = \frac{n!}{n+1}\sum_{j=1}^{n}\sum_{r=0}^{n-j} \frac{(-1)^r}{(n-j)!(j-1)!}\binom{n+1}{j}\binom{n-j+1}{r}$$

$$\cdot \int_0^1 e^{t(j+r+(n+1-j-r)z)/(n+1)}\left(1-\frac{j+r+(n+1-j-r)z}{n+1}\right)^{j-1}$$

$$\cdot \left(\frac{j+r+(n+1-j-r)z-j-r}{n+1}\right)^{n-j}(n+1-j-r)\;dz$$

$$= \frac{n!}{n+1}\sum_{j=1}^{n}\sum_{r=0}^{n-j} \frac{(-1)^r}{(n-j)!(j-1)!}\binom{n+1}{j}\binom{n-j+1}{r} e^{t(j+r)/(n+1)}$$

$$\cdot \frac{(n+1-j-r)^n}{(n+1)^{n-1}}\int_0^1 e^{t\left(1-\frac{j+r}{n+1}\right)z} z^{n-j}(1-z)^{j-1}\,dz$$

$$= n \sum_{j=1}^{n} \sum_{r=0}^{n-j} (-1)^r \frac{(n-1)!}{(n-j)!(j-1)!} \frac{n!}{j!(n+1-j)!} \frac{(n+1-j)!}{r!(n+1-j-r)!}$$

$$\cdot\ e^{t(j+r)/(n+1)} (n+1-j-r)(1 - \frac{j+r}{n+1})^{n-1}$$

$$\cdot \int_0^1 e^{t(1 - \frac{j+r}{n+1})z} z^{n-j} (1-z)^{j-1} dz$$

$$= n \sum_{j=1}^{n} \sum_{r=0}^{n-j} (-1)^r \binom{n-1}{j-1} \binom{n}{j} \binom{n-j}{r} e^{t(j+r)/(n+1)} (1 - \frac{j+r}{n+1})^{n-1}$$

$$\cdot \int_0^1 e^{t(1 - \frac{j+r}{n+1})z} z^{n-j} (1-z)^{j-1} dz \quad .$$

Let $(1-z) = q$, $dz = -dq$. Then

$$(2.2.4) \qquad M_n(t) = n \sum_{j=1}^{n} \sum_{r=0}^{n-j} (-1)^r \binom{n-1}{j-1} \binom{n}{j} \binom{n-j}{r} e^{t(j+r)/(n+1)} (1 - \frac{j+r}{n+1})^{n-1}$$

$$\cdot \int_0^1 e^{t(1 - \frac{j+r}{n+1})(1-q)} (1-q)^{n-j} q^{j-1} dq \ .$$

Consider the standardized variable $\frac{\mu(X)-\mu_n}{\sigma_n}$ with moment generating function

$$\psi_n(t) = E[e^{((\mu(X)-\mu_n)/\sigma_n)t}] = e^{-\mu_n t/\sigma_n} E(e^{\mu(X)t/\sigma_n})$$

$$= e^{-\mu_n t/\sigma_n} M_n(t/\sigma_n) \quad .$$

From (2.2.4)

$$\psi_n(t) = n\, e^{(t/\sigma_n)-(\mu_n t/\sigma_n)} \int_0^1 \sum_{j=1}^{n} \binom{n-1}{j-1}\binom{n}{j} (1-q)^{n-j} q^{j-1}$$

$$\cdot\, e^{-(tq/\sigma_n)+(tqj/(\sigma_n(n+1)))} \sum_{r=0}^{n-j} (-1)^r \binom{n-j}{r} e^{tqr/((n+1)\sigma_n)}$$

$$\cdot \left(1 - \frac{j+r}{n+1}\right)^{n-1} dq \; .$$

Let

$$\text{(2.2.5)} \quad \begin{cases} \left(1 - \dfrac{y}{n+1}\right)^{n-1} = e^{-y} \displaystyle\sum_{k=0}^{\infty} a_k(n)\, y^k \quad \text{so that} \\ \displaystyle\sum_{k=0}^{\infty} a_k(n)\, y^k = e^y \left(1 - \dfrac{y}{n+1}\right)^{n-1} . \end{cases}$$

For $y = j+r$, this yields

$$\left(1 - \frac{j+r}{n+1}\right)^{n-1} = e^{-(j+r)} \sum_{k=0}^{\infty} a_k(n)\, (j+r)^k$$

and, since

$$(j+r)^k = \frac{d^k}{dp^k} \left(e^{(j+r)p}\right)\Big|_{p=0} \; ,$$

we have

$$\left(1 - \frac{j+r}{n+1}\right)^{n-1} = e^{-(j+r)} \sum_{k=0}^{\infty} a_k(n) \frac{d^k}{dp^k} \left(e^{(j+r)p}\right)\Big|_{p=0} \; . \,^{*}$$

Substituting this into the expression for $\psi_n(t)$, we get

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*This method of dealing with the term $\left(1 - \frac{j+r}{n+1}\right)^{n-1}$ was suggested by Dr. J.R. McGregor.

$$\psi_n(t) = \sum_{k=0}^{\infty} a_k(n)\, \psi_n^k(t,0) \quad , \tag{2.2.6}$$

where

$$\psi_n^k(t,0) = \frac{d^k}{dp^k} \Psi_n(t,p)\Big|_{p=0} \quad ,$$

and

$$\Psi_n(t,p) = ne^{t(1-\mu_n)/\sigma_n} \int_0^1 e^{-tq/\sigma_n} \tag{2.2.7}$$

$$\cdot \sum_{j=1}^{n} \binom{n-1}{j-1} \binom{n}{j} (1-q)^{n-j}\, q^{j-1}\, e^{tqj/((n+1)\sigma_n)}$$

$$\cdot \sum_{r=0}^{n-j} (-1)^r \binom{n-j}{r} e^{tqr/((n+1)\sigma_n) - (j+r) + p(j+r)} \, dq$$

$$= ne^{t(1-\mu_n)/\sigma_n} \int_0^1 e^{-tq/\sigma_n} \sum_{j=1}^{n} \binom{n-1}{j-1} \binom{n}{j} (1-q)^{n-j}\, q^{j-1}$$

$$\cdot \left\{e^{tq/((n+1)\sigma_n) + p-1}\right\}^j$$

$$\cdot \sum_{r=0}^{n-j} (-1)^r \binom{n-j}{r} \left\{e^{tq/((n+1)\sigma_n) + p-1}\right\}^r dq$$

$$= ne^{t(1-\mu_n)/\sigma_n} \int_0^1 e^{-tq/\sigma_n}$$

$$\cdot \sum_{j=1}^{n} \binom{n-1}{j-1} \binom{n}{j} (1-q)^{n-j}\, q^{j-1}\, \beta^j\, (1-\beta)^{n-j} \, dq \quad ,$$

where

$$\beta = e^{tq/((n+1)\sigma_n) + p-1} \quad . \tag{2.2.8}$$

Thus

$$\text{(2.2.9)} \qquad \Psi_n(t,p) = ne^{t(1-\mu_n)/\sigma_n} \int_0^1 e^{-tq/\sigma_n} \beta$$

$$\cdot \sum_{j=1}^{n} \binom{n-1}{j-1} \binom{n}{j} (q\beta)^{j-1} \{(1-q)(1-\beta)\}^{n-j} \, dq \ .$$

Consider

$$S_1 = \sum_{j=1}^{n} \binom{n-1}{j-1} \binom{n}{j} (q\beta)^{j-1} \{(1-q)(1-\beta)\}^{n-j}$$

$$= \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \binom{n}{\nu+1} (q\beta)^{\nu} \{(1-q)(1-\beta)\}^{n-1-\nu}$$

$$= \{(1-q)(1-\beta)\}^{n-1} \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \binom{n-1+1}{n-1-\nu} \left\{\frac{q\beta}{(1-q)(1-\beta)}\right\}^{\nu}.$$

Let

$$\frac{q\beta}{(1-q)(1-\beta)} = \frac{s-1}{s+1} \qquad \text{where} \qquad s = \frac{q\beta+(1-q)(1-\beta)}{(1-q)(1-\beta)-q\beta} \quad .$$

Then

$$S_1 = \{(1-q)(1-\beta)\}^{n-1} \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \binom{n-1+1}{n-1-\nu} \left\{\frac{(s-1)/2}{(s+1)/2}\right\}^{\nu}$$

$$= \frac{\{(1-q)(1-\beta)\}^{n-1}}{(s+1/2)^{n-1}} \sum_{\nu=0}^{n-1} \binom{n-1}{\nu} \binom{n-1+1}{n-1-\nu} \left(\frac{s-1}{2}\right)^{\nu} \left(\frac{s+1}{2}\right)^{n-1-\nu}$$

$$= \left\{\frac{2(1-q)(1-\beta)}{s+1}\right\}^{n-1} P_{n-1}^{(1,0)}(s) \quad ,$$

where $P_{n-1}^{(1,0)}(s)$ is the Jacobi polynomial with parameters $\alpha = 1$, $\beta = 0$ (see Szegö (1974), equation (4.3.2)). Now

$$s = \frac{q\beta+(1-q)(1-\beta)}{(1-q)(1-\beta)-q\beta} = \frac{q\beta+(1-q)(1-\beta)}{1-q-\beta} ,$$

so that

$$s+1 = \frac{q\beta+1-q-\beta+q\beta+1-q-\beta}{1-q-\beta} = \frac{2(1-q)(1-\beta)}{1-q-\beta}$$

giving

$$S_1 = (1-q-\beta)^{n-1} P_{n-1}^{(1,0)} \left(\frac{(1-q)(1-\beta)+q\beta}{1-q-\beta}\right) . \tag{2.2.10}$$

Consider the following theorem of Darboux which is given in Szegö (1974, page 196).

Theorem 2.2.1: Let α and β be arbitrary real numbers. Then

$$P_n^{(\alpha,\beta)}(x) \cong (x-1)^{-\alpha/2} (x+1)^{-\beta/2} \{(x+1)^{1/2} + (x-1)^{1/2}\}^{\alpha+\beta}$$
$$\cdot (2\pi n)^{-1/2} (x^2-1)^{-1/4} \{x+(x^2-1)^{1/2}\}^{n+1/2} ,$$

where x is outside of the closed interval $[-1,1]$. This formula holds uniformly in the exterior of an arbitrary closed curve which encloses the segment $[-1,1]$, in the sense that the ratio tends uniformly to 1.

Lemma 2.2.1: $s = \dfrac{(1-q)(1-\beta)+q\beta}{1-q-\beta}$ lies outside the interval $[-1,1]$.

Proof: Recall that $\beta = e^{tq/((n+1)\sigma_n) + p-1}$, $0 < q < 1$ and also β need only be differentiated with respect to p and t in a neighbourhood of $p = 0$, $t = 0$. Therefore, we may take $-\epsilon < t < \epsilon$ and $-\delta < p < \delta$, and choose ϵ and δ arbitrarily small. Thus

$$-\frac{\epsilon q}{(n+1)\sigma_n} - \delta - 1 < \frac{tq}{(n+1)\sigma_n} + p - 1 < \frac{\epsilon q}{(n+1)\sigma_n} + \delta - 1$$

or

$$e^{-1-(\epsilon q/((n+1)\sigma_n)+\delta)} < \beta < e^{-1+(\epsilon q/((n+1)\sigma_n)+\delta)} .$$

Thus $e^{-1} < 1/2$ implies that we can choose ϵ and δ so that $0 < \beta < 1/2$. Also, since $0 < q < 1$, there exists q_0 such that $q_0 = 1-\beta$. Now

$$\frac{d}{dq}\left[\frac{(1-q)(1-\beta)+q\beta}{1-q-\beta}\right] = \frac{2\beta}{(1-\beta-q)^2}\left[1 - \beta + \frac{q(1-q)t}{(n+1)\sigma_n}\right] .$$

By restricting t to an open interval $(-\epsilon,\epsilon)$, $\epsilon > 0$ of arbitrarily small width, the derivative is positive over the range of q except when the function itself is not defined at $q = 1-\beta$. Now at,

$$q = 0, \quad \frac{(1-q)(1-\beta)+q\beta}{1-q-\beta} = 1 \quad \text{and} \quad q = 1, \quad \frac{(1-q)(1-\beta)+q\beta}{1-q-\beta} = -1 .$$

Therefore s lies outside the interval $[-1,+1]$ when $q \in (0,1)$. Thus Theorem 2.2.1 is applicable. Note that

$$s-1 = \frac{q\beta+1-q-\beta+q\beta-1+q+\beta}{1-q-\beta} = \frac{2q\beta}{1-q-\beta}$$

so that

$$s^2-1 = \frac{4q\beta(1-q)(1-\beta)}{(1-q-\beta)^2} .$$

Then for large n,

$$P_{n-1}^{(1,0)}\left(\frac{(1-q)(1-\beta)+q\beta}{1-q-\beta}\right) \cong \left(\frac{2q\beta}{1-q-\beta}\right)^{-1/2} \left[\left\{\frac{2(1-q)(1-\beta)}{1-q-\beta}\right\}^{1/2} + \left\{\frac{2q\beta}{1-q-\beta}\right\}^{1/2}\right]$$

$$\cdot\ [2\pi(n-1)]^{-1/2} \left[\frac{4q\beta(1-q)(1-\beta)}{(1-q-\beta)^2}\right]^{-1/4}$$

$$\cdot \left\{\frac{q\beta+(1-q)(1-\beta)}{1-q-\beta} + \frac{2q^{1/2}\beta^{1/2}(1-q)^{1/2}(1-\beta)^{1/2}}{1-q-\beta}\right\}^{n-1+(1/2)}$$

$$= \frac{(1-q-\beta)^{1/2}}{\sqrt{2}\, q^{1/2}\beta^{1/2}} \cdot \frac{1}{\sqrt{2}\, \pi^{1/2}(n-1)^{1/2}} \cdot \frac{(1-q-\beta)^{1/2}}{\sqrt{2}\, q^{1/4}\beta^{1/4}(1-q)^{1/4}(1-\beta)^{1/4}}$$

$$\cdot \frac{\sqrt{2}}{(1-q-\beta)^{1/2}} \{q^{1/2}\beta^{1/2} + (1-q)^{1/2}(1-\beta)^{1/2}\}$$

$$\cdot \frac{1}{(1-q-\beta)^{n-1/2}} \{[q^{1/2}\beta^{1/2} + (1-q)^{1/2}(1-\beta)^{1/2}]^2\}^{n-1/2}$$

$$= \frac{\{q^{1/2}\beta^{1/2} + (1-q)^{1/2}(1-\beta)^{1/2}\}^{2n}}{2\{(n-1)\pi\}^{1/2}(1-q-\beta)^{n-1}\, q^{3/4}\beta^{3/4}(1-q)^{1/4}(1-\beta)^{1/4}}\ .$$

Substituting in (2.2.10) we get

$$S_1 \cong \frac{\{q^{1/2}\beta^{1/2} + (1-q)^{1/2}(1-\beta)^{1/2}\}^{2n}}{2\{(n-1)\pi\}^{1/2}\, q^{3/4}\beta^{3/4}(1-q)^{1/4}(1-\beta)^{1/4}}$$

which when substituted in (2.2.9) yields

(2.2.11) $$\Psi_n(t,p) \cong \frac{ne^{t(1-\mu_n)/\sigma_n}}{2\{(n-1)\pi\}^{1/2}}$$

$$\cdot \int_0^1 e^{-tq/\sigma_n} \frac{\{q^{1/2}\beta^{1/2}+(1-q)^{1/2}(1-\beta)^{1/2}\}^{2n}}{q^{3/4}(1-q)^{1/4}} \left(\frac{\beta}{1-\beta}\right)^{1/4} dq.$$

To evaluate this integral, we use the following generalization of Laplace's method (see Olver, 1974, page 333):

<u>Theorem 2.2.2:</u> Let

$$I(x) = \int_0^k \exp\{-xp(t)+x^{\nu/\mu} r(t)\}q(t)\,dt \quad ;$$

where k is fixed and positive, and assume that:

(i) In the interval $(0,k]$, $p'(t)$ is continuous and positive, and the real or complex functions $q(t)$ and $r(t)$ are continuous.

(ii) As $t \to 0+$,

$$p(t) = p(0) + P\,t^{\mu} + o\left(t^{\mu_1}\right) \quad ;$$

$$p'(t) = \mu\,P\,t^{\mu-1} + o\left(t^{\mu_1-1}\right) \quad ;$$

$$q(t) = Q\,t^{\lambda-1} + o\left(t^{\lambda_1-1}\right) \quad ;$$

$$r(t) = R\,t^{\nu} + o\left(t^{\nu_1}\right) \quad ; \quad \text{where } P > 0 ,$$

$$\mu_1 > \mu > 0, \quad \lambda_1 > \lambda > 0, \quad \mu > \nu \geq 0, \quad \text{and} \quad \nu_1 > \nu.$$

Then

$$I(x) = \frac{Q}{\mu} F_i\left(\frac{\nu}{\mu}, \frac{\lambda}{\mu}; \frac{R}{P^{\nu/\mu}}\right)\frac{e^{-xp(0)}}{(Px)^{\lambda/\mu}}\left\{1 + o\left(\frac{1}{x^{\tilde{w}/\mu}}\right)\right\}$$

as $x \to \infty$, where

$$\tilde{w} = \min\,(\lambda_1-\lambda,\ \mu_1-\mu,\ \nu_1-\nu) \quad \text{and}$$

$F_i(\alpha,\beta;y)$ is the Faxen's integral

$$F_i(\alpha,\beta;y) = \int_0^\infty \exp\{-\tau+y\tau^\alpha\}\tau^{\beta-1}\,d\tau ,$$

$$(0 \leq \text{Re}\ \alpha < 1,\ \text{Re}\ \beta > 0) .$$

In particular

$$F_i(\tfrac{1}{2}, \tfrac{1}{2};y) + F_i(\tfrac{1}{2}, \tfrac{1}{2};-y) = 2\pi^{1/2} \exp(\tfrac{1}{4} y^2) .$$

To apply this theorem to the integral in (2.2.11), note that

(i) $\lim_{n \to \infty} \mu_n = 1 - e^{-1}$,

(ii) $\lim_{n \to \infty} (n+2)\,\sigma_n^2 = 2e^{-1} - 5e^{-2}$ so that $\sigma_n^2 = \dfrac{k_n^2}{n}$

where $k_n^2 \to 2e^{-1} - 5e^{-2}$, and

(iii) $\beta = e^{tq/((n+1)\sigma_n) + p-1}$ so that for large n, $\beta \cong e^{p-1}$.

Then (2.2.11) reduces to

$$\Psi_n(t,p) \approx \frac{ne^{te^{-1}\sqrt{n}/k_n}}{2\{(n-1)\pi\}^{1/2}} \int_0^1 e^{-\sqrt{n}\ tq/k_n} \frac{e^{(p-1)/4}}{q^{3/4}(1-q)^{1/4}(1-e^{p-1})^{1/4}}$$

$$\cdot \{q^{1/2}\ e^{(p-1)/2}+(1-q)^{1/2}(1-e^{p-1})^{1/2}\}^{2n}\, dq$$

$$= \frac{ne^{(p-1)/4}}{2\{(n-1)\pi\}^{1/2}(1-e^{p-1})^{1/4}} \int_0^1 \frac{e^{(\sqrt{n}\ t/k_n)(e^{-1}-q)}}{q^{3/4}(1-q)^{1/4}}\ \theta^{2n}\, dq , \qquad (2.2.12)$$

where

$$\theta(q,p) \equiv \theta = q^{1/2}\ e^{(p-1)/2} + (1-q)^{1/2}(1-e^{p-1})^{1/2} .$$

Let

$$I_n = \int_0^1 e^{(\sqrt{n}t/k_n)(e^{-1}-q)} \frac{\theta^{2n}}{q^{3/4}(1-q)^{1/4}}\, dq$$

$$= \int_0^1 \exp\{-n(-2\ell n\ \theta) + n^{1/2} \frac{t}{k_n} (e^{-1}-q)\}(q^{3/4}(1-q)^{1/4})^{-1}\, dq\ .$$

Note that $\theta(0,p) = (1-e^{p-1})^{1/2}$, $\theta(1,p) = e^{(p-1)/2}$ and $\theta(e^{p-1},p) = e^{p-1} + 1 - e^{p-1} = 1$. Also,

$$\frac{d}{dq}\ \theta(q,p) = \frac{1}{2}\, q^{-1/2}\, e^{(p-1)/2} - \frac{1}{2}(1-q)^{-1/2}(1-e^{p-1})^{1/2}$$

so that $\frac{d}{dq}\ \theta(e^{p-1},p) = 0$. Let $y = q - e^{p-1}$ so that when $q = 0$, $y = -e^{p-1}$ and when $q = 1$, $y = 1 - e^{p-1}$. Thus

$$I_n = e^{(\sqrt{n}t/k_n)(e^{-1}-e^{p-1})} \{ \int_{-e^{p-1}}^{0} + \int_0^{1-e^{p-1}} \}$$

$$\exp\{-n(-2\ell n\ \theta(y+e^{p-1},p)) + n^{1/2} \frac{t}{k_n} (-y)\}$$

$$[(y+e^{p-1})^{3/4}\ (1-y-e^{p-1})^{1/4}]^{-1}\ dy$$

$$= I_{n_1} + I_{n_2}\ .$$

Consider I_{n_1} and put $w = -y$. Then

$$I_{n_1} = e^{(\sqrt{n}t/k_n)(e^{-1}-e^{p-1})} \int_0^{e^{p-1}} \exp\{-np(w) + n^{1/2} r(w)\} q(w)\, dw\ ,$$

where

$$p(w) = -2\ \ln\ \{(1+w-e^{p-1})^{1/2}\ (1-e^{p-1})^{1/2} + (e^{p-1}-w)^{1/2}\ e^{(p-1)/2}\}\ ,$$

$$r(w) = \frac{t}{k_n}\, w \qquad \text{and} \qquad q(w) = [(e^{p-1}-w)^{3/4}\ (1+w-e^{p-1})^{1/4}]^{-1}\ .$$

Compare with Theorem 2.2.2, then $\frac{\nu}{\mu} = \frac{1}{2}$ and also as $r(w) = \frac{t}{k_n} w$, this implies $R = \frac{t}{k_n}$ and $\nu = 1$ so that $\mu = 2$. Now to find the value of P, note that $\frac{p(w) - p(0)}{w^2} \to P$ as $w \to 0+$ and also $p(0) = 0$. Thus

$$\lim_{w \to 0+} \frac{p(w) - p(0)}{w^2} = \lim_{w \to 0+} \frac{p(w)}{w^2} = \lim_{w \to 0+} \frac{p'(w)}{2w}\ .$$

Now

$$p'(w) = \frac{-2\{\frac{1}{2}(1+w-e^{p-1})^{-1/2}(1-e^{p-1})^{1/2} - \frac{1}{2}(e^{(p-1)}-w)^{-1/2}e^{(p-1)/2}\}}{(1+w-e^{p-1})^{1/2}(1-e^{p-1})^{1/2} + (e^{p-1}-w)^{1/2}e^{(p-1)/2}}$$

and so $p'(0) = 0$. Therefore

$$P = \lim_{w \to 0+} \frac{p''(w)}{2} = \frac{1}{4}\ \{(1-e^{p-1})^{-1/2} + e^{-(p-1)/2}$$

$$= \frac{1}{4\{e^{p-1}(1-e^{p-1})\}}$$

Also

$$\lim_{w \to 0+} \frac{q(w)}{w^{\lambda-1}} = Q\ .$$

Let $\lambda = 1$, then $\lim_{w \to 0+} q(w) = Q$ which implies

$$Q = [e^{3(p-1)/4}\ (1-e^{p-1})^{1/4}]^{-1}\ .$$

Now applying Theorem (2.2.2),

$$I_{n_1} \cong e^{(\sqrt{n}t/k_n)(e^{-1}-e^{p-1})} \frac{Q}{2(nP)^{1/2}} F_i(\tfrac{1}{2},\tfrac{1}{2};\ \frac{R}{P^{1/2}}) \ .$$

Similarly,

$$I_{n_2} \cong e^{(\sqrt{n}t/k_n)(e^{-1}-e^{p-1})} \frac{Q}{2(nP)^{1/2}} F_i(\tfrac{1}{2},\tfrac{1}{2};\ -\frac{R}{P^{1/2}}) \ .$$

Thus

$$I_n = I_{n_1} + I_{n_2}$$

$$\cong e^{(\sqrt{n}t/k_n)(e^{-1}-e^{p-1})} \frac{Q}{2(nP)^{1/2}} \cdot 2\pi^{1/2} \exp\left(\frac{R^2}{4P}\right) \ .$$

Substituting this into (2.2.12), we get

$$\Psi_n(t,p) \cong \frac{n\ e^{(p-1)/4}}{2\{(n-1)\pi\}^{1/2}(1-e^{p-1})^{1/4}}\ e^{(\sqrt{n}t/k_n)(e^{-1}-e^{p-1})}$$

$$\cdot \frac{2\pi^{1/2}\ 2\ e^{(p-1)/2}(1-e^{p-1})^{1/2}}{2n^{1/2}\ e^{3(p-1)/4}(1-e^{p-1})^{1/4}} \exp\left\{\frac{t^2}{k_n^2}\ e^{p-1}\ (1-e^{p-1})\right\}$$

$$\approx e^{(\sqrt{n}t/k_n)(e^{-1}-e^{p-1}) + t^2/k_n^2\ e^{p-1}(1-e^{p-1})}$$

$$= \exp\{g(p)\} \ ,$$

where

$$g(p) = \frac{\sqrt{n}\ t}{k_n}\ (e^{-1}-e^{p-1}) + \frac{t^2}{k_n^2}\ e^{p-1}\ (1-e^{p-1}) \ .$$

Now

$$\frac{d}{dp}\Psi_n(t,p) \approx e^{g(p)}\, g'(p) \ ,$$

$$\frac{d^2}{dp^2}\Psi_n(t,p) \approx e^{g(p)}\, \{(g'(p))^2 + g''(p)\} \ ,$$

$$\frac{d^3}{dp^3}\Psi_n(t,p) \approx e^{g(p)}\, \{(g'(p))^3 + 3g'(p)g''(p)+g'''(p)\}$$

$$\cdots \qquad\qquad \cdots$$

$$\cdots \qquad\qquad \cdots$$

$$\frac{d^k}{dp^k}\Psi_n(t,p) \approx e^{g(p)}\{(g'(p))^k + (2k-3)(g'(p))^{k-2}g''(p) + \ldots\} \ .$$

Now

$$g'(p) = -\frac{\sqrt{n}t\ e^{p-1}}{k_n} + \frac{t^2}{k_n^2}\{e^{p-1}(1-e^{p-1})-e^{2(p-1)}\}$$

and

$$g''(p) = -\frac{\sqrt{n}t\ e^{p-1}}{k_n} + \frac{t^2}{k_n^2}\{e^{p-1}(1-e^{p-1})-3\ e^{2(p-1)}\} \ .$$

In general, we shall have

$$\frac{d^k}{dp^k}\Psi_n(t,0) \approx e^{(t^2/k_n^2)e^{-1}(1-e^{-1})}\{(-\frac{\sqrt{n}t\ e^{-1}}{k_n})^k\}\ (1 + O(n^{-1/2})) \ .$$

Substituting this into (2.2.6) we get

$$\psi_n(t) \approx \sum_{k=0}^{\infty} a_k(n)\ e^{(t^2/k_n^2)e^{-1}(1-e^{-1})}\ (-\frac{\sqrt{n}t\ e^{-1}}{k_n})^k \ ,$$

which by (2.2.5) is

$$= e^{(t^2/k_n^2)e^{-1}(1-e^{-1})} e^{-\sqrt{n}te^{-1}/k_n} \left(1 + \frac{\sqrt{n}te^{-1}}{k_n(n+1)}\right)^{n-1}$$

$$= e^{(t^2/k_n^2)e^{-1}(1-e^{-1})} e^{-\sqrt{n}te^{-1}/k_n} e^{(n-1)\ln(1+(\sqrt{n}te^{-1})/(k_n(n+1)))}$$

$$= e^{(t^2/k_n^2)e^{-1}(1-e^{-1})} e^{-\sqrt{n}te^{-1}/k_n}$$

$$\cdot\ e^{(n-1)[\sqrt{n}te^{-1}/(k_n(n+1)) - \frac{1}{2} nt^2e^{-1}/(k_n^2(n+2)^2) + \ldots]}$$

$$\cong e^{(t^2/k_n^2)e^{-1}(1-e^{-1})} e^{-\sqrt{n}te^{-1}/k_n}$$

$$\cdot\ e^{[\sqrt{n}te^{-1}/k_n - t^2e^{-2}/2k_n^2]\{1+O(n^{-1})\}}$$

$$\cong e^{(t^2/k_n^2)\{e^{-1}-e^{-2} - \frac{1}{2} e^{-2}\}} = e^{(t^2/2k_n^2)(2e^{-1}-3e^{-2})} .$$

Note that $k_n^2 \to 2e^{-1}-5e^{-2}$ as n goes to infinity. Hence

$$\lim_{n \to \infty} \psi_n(t) = e^{(t^2/2)\{(2e^{-1}-3e^{-2})/(2e^{-1}-5e^{-2})\}} .$$

Since for every n, $\psi_n(t)$ is the moment generating function of a random variable with zero mean and unit variance the same should be true of $\lim_{n \to \infty} \psi_n(t)$. It is clear, therefore, that at some stage the approximations used have been unsatisfactory since the result obtained above implies that the limiting distribution is normal with zero mean and variance $\{(2e^{-1}-3e^{-2})/(2e^{-1}-5e^{-2})\}$ instead of unity. We have not been able to refine the approximations to eliminate this discrepancy. It is perhaps of some interest to

note that the asymptotic evaluations of integrals by modifications of the Laplace method, such as we have used here, often result in functions which are exact except for the constants involved. Thus, for example, if one applies the method of Steepest Descents to determine the F-distribution the result is exact except that the factorials in the normalizing constant are replaced by their Stirling's approximations. This may provide a clue of where one should look to refine the analysis to produce the exact result.

2.3 Moments of the Variable $\{1-\mu(X)\}$

For $D = 1/(n+1)$, this section contains a result about the non-central moments of the random variable $(1-\mu(X))$.

Theorem 2.3.1: The k^{th} non-central moment of the variable $\{1-\mu(X)\}$ when $D = 1/(n+1)$ is given by

$$m_k = \frac{1}{\binom{n+k}{k}} \sum_{i=1}^{k} \binom{n+1}{i}\binom{k-1}{i-1}\left(1 - \frac{i}{n+1}\right)^{n+k} \tag{2.3.1}$$

$$k = 1,2,\ldots,n+1 \quad .$$

Proof:

$$m_k = E(1-\mu(X))^k$$

$$= \int_{1/n+1}^{1} (1-x)^k f_n(x)\,dx$$

$$= \sum_{\ell=1}^{n} \int_{\ell/(n+1)}^{(\ell+1)/(n+1)} (1-x)^k f_n(x)\,dx$$

$$= \sum_{\ell=1}^{n} \int_{\ell/(n+1)}^{(\ell+1)/(n+1)} n! \sum_{j=1}^{\ell} \sum_{r=0}^{\ell-j} \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j} \binom{n-j+1}{r}$$

$$\cdot\ (1-x)^{k+j-1} \left[x - \frac{j+r}{n+1}\right]^{n-j} dx \quad .$$

Comparing with (2.2.2) and following the same analysis as done there, we get

$$m_k = \frac{n!}{(n+1)} \sum_{j=1}^{n} \sum_{r=0}^{n-j} \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j} \binom{n-j+1}{r} \tag{2.3.2}$$

$$\cdot \int_{j+r}^{n+1} \left(1 - \frac{y}{n+1}\right)^{k+j-1} \left(\frac{y-j-r}{n+1}\right)^{n-j} dy \quad .$$

Let

$$I = \int_{(j+r)}^{(n+1)} \left(1 - \frac{y}{n+1}\right)^{k+j-1} \left(\frac{y-j-r}{n+1}\right)^{n-j} dy$$

and take $z = \frac{y-j-r}{n+1-j-r}$ so that $0 \le z \le 1$ and $dy = (n+1-j-r)dz$.

Then

$$I = \int_0^1 \left(1 - \frac{j+r+(n+1-j-r)z}{n+1}\right)^{k+j-1}$$

$$\cdot \left(\frac{j+r+(n+1-j-r)z-j-r}{n+1}\right)^{n-j} (n+1-j-r)dz$$

$$= \frac{(n+1-j-r)^{n+k}}{(n+1)^{n+k-1}} \int_0^1 (1-z)^{k+j-1} z^{n-j} dz$$

$$= (n+1)\left(1 - \frac{j+r}{n+1}\right)^{n+k} \beta(n-j+1,k+j)$$

$$= (n+1)\left(1 - \frac{j+r}{n+1}\right)^{n+k} \frac{(n-j)!(j+k-1)!}{(n+k)!} \quad .$$

Substituting in (2.3.2) we get

$$m_k = \frac{n!}{(n+1)} \sum_{j=1}^{n} \sum_{r=0}^{n-j} \frac{(-1)^r}{(n-j)!(j-1)!} \binom{n+1}{j} \binom{n-j+1}{r} \left(1 - \frac{j+r}{n+1}\right)^{n+k}$$

$$\cdot (n+1) \frac{(n-j)!(j+k-1)!}{(n+k)!} \quad .$$

Put $(j+r) = i$, then

$$m_k = \frac{n!}{(n+k)!} \sum_{j=1}^{n} \sum_{i=j}^{n} \frac{(-1)^{i-j}}{(j-1)!} \frac{(n+1)!}{j!(n+1-j)!} \frac{(n+1-j)!}{(i-j)!(n-i+1)!}$$

$$\cdot (k+j-1)! \left(1 - \frac{i}{n+1}\right)^{n+k}$$

$$= \frac{n!\ k!}{(n+k)!} \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{i-j} \left\{\frac{(k+j-1)!}{(j-1)!k!}\right\} \left\{\frac{(n+1)!}{i!(n+1-i)!}\right\}$$

$$\cdot \left\{\frac{i!}{j!(i-j)!}\right\} \left(1 - \frac{i}{n+1}\right)^{n+k} \quad .$$

Thus

$$(2.3.3) \qquad m_k = \frac{1}{\binom{n+k}{k}} \sum_{i=1}^{n} \sum_{j=1}^{i} (-1)^{i-j} \binom{k+j-1}{j-1} \binom{i}{j}$$

$$\cdot \binom{n+1}{i} \left(1 - \frac{i}{n+1}\right)^{n+k} \quad .$$

Consider the sum

$$\sum_{j=1}^{i} (-1)^{i-j} \binom{k+j-1}{j-1} \binom{i}{j} = \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} \binom{k+j-1}{j-1} \quad .$$

Compare this with the identity

$$\binom{n+p}{m+p} = \sum_{k_1=0}^{m} (-1)^{k_1+m} \binom{m}{k_1} \binom{n+p+k_1}{p+k_1}$$

(see Riordan (1968), page 11). Thus identifying the variables in the two sums, we get

$$\binom{k-1}{i-1} = \sum_{j=0}^{i} (-1)^{i+j} \binom{i}{j} \binom{k+j-1}{j-1} \quad .$$

Thus (2.3.3) reduces to

$$m_k = \frac{1}{\binom{n+k}{k}} \sum_{i=1}^{n} \binom{n+1}{i} \binom{k-1}{i-1} \left(1 - \frac{i}{n+1}\right)^{n+k}$$

$$= \frac{1}{\binom{n+k}{k}} \sum_{i=1}^{k} \binom{n+1}{i} \binom{k-1}{i-1} \left(1 - \frac{i}{n+1}\right)^{n+k} \quad ,$$

$$k = 1,2,\ldots,n+1 \ .$$

2.4 Plots and Tables

The results of the previous section strongly suggest that the limiting distribution of the standardized random variable $\mu(X)$ is standard normal when $D = 1/(n+1)$. In this section we examine the probability function, $f_n(x)$, of the random variable $\mu(X)$ by plotting it for four values of n. For each value of n, three values of D are considered: one less than, one equal to and one greater than $1/(n+1)$. Appendix I contains the programs used for generating the data, Appendix II contains the data generated and Appendix III

contains the plots obtained by using that data. In plots 2, 3 and 4 the scaling used for the Y axis is 1 unit = 2.0.

Looking at the plots in Appendix III, it is clear that the function is symmetric only when $D = 1/(n+1)$ and is skewed in other cases. Also, even for the small values of n considered, the function tends to be a spike as n increases indicating that the variance decreases very rapidly.

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APPENDIX I

PROGRAM I

```
C ***************************************************************
C  THIS PROGRAM FINDS POINTS FOR THE PROBABILITY FUNCTION FNX.*
C  GIVEN THE VALUE OF N+2 AND THREE VALUES OF D AS INPUT, IT   *
C  PRINTS THE VALUES OF FNX FOR EACH VALUE OF D.INCREMENT IN   *
C  X IS .02.                                                    *
C***************************************************************
C
C
C
C***************************************************************
C                                                               *
C  SUBROUTINE COMB USES THE METHOD OF PASCAL'S TRIANGLE TO      *
C  CALCULATE N CHOOSE M FOR VARIOUS VALUES OF N AND M. A(I,J)  *
C  GIVES THE VALUE OF I-1 CHOOSE J-1 FOR   J < OR = I.          *
C                                                               *
C***************************************************************
C
C
C
         SUBROUTINE COMB(N2,A)
         DIMENSION A(N2,N2)
         DO 5 I=1,N2
         A(I,1)=1
         A(I,I)=1
   5     CONTINUE
         N1=N2-1
         DO 10 J=2,N1
         J1=J+1
         DO 10 I=J1,N2
         A(I,J)=A(I-1,J)+A(I-1,J-1)
   10    CONTINUE
         RETURN
         END
C
C
C
C***************************************************************
C                                                              *
C  GIVEN THE MATRIX OF COMBINATIONS,VALUE OF N+2 AND A VALUE *
C  OF D, THIS SUBROUTINE CALCULATES AND PRINTS THE VALUES OF *
C  THE PROBABILITY FUNCTION FNX. INCREMENT IN X IS .02. THE   *
```

```
C  LAST VALUE PRINTED IS (0.0,0.0).                                    *
C                                                                      *
C***********************************************************************
C
C
C
         SUBROUTINE POINTS(N2,D,A)
         DIMENSION A(N2,N2)
         REAL M1
         INTEGER Q,R
         B1=0
         B2=0
         M=INT(1/D)
         N1=N2-1
         N=N1-1
         IF (N .LT. M)  M=N
         M1=N1*D
         IF (M1 .GT. 1)  M1=1
         DO 15 Q=1,M
         X=Q*D
         X2=AMIN1(M1,(Q+1)*D)
 2       IF (X .LT. X2) GOTO 5
         GOTO 15
 5       FNX=0
         DO 10 J=1,Q
         JQ1=Q-J+1
         DO 10 R=1,JQ1
         S=A(N1,J+1)*A(N2,J+1)*A(N2-J,R)*J
   *     *(-1)**(R-1)*(1-X)**(J-1)
         IF( ((X-D*(J+R-1)) .EQ. (-.0)) .AND.
   *     (N .EQ. J)) GOTO 10
         S=S*(X-D*(J+R-1))**(N-J)
 7       FNX=FNX+S
 10      CONTINUE
         WRITE(6,100) X,FNX
 100     FORMAT(' ',F12.8,5X,F12.8)
         X=X+.02
         GOTO 2
 15      CONTINUE
         WRITE(6,100) B1,B2
         RETURN
```

```
      END
C
C
C
C  MAIN PROGRAM STARTS HERE.  DIMENSION OF A IS TO BE CHANGED*
C  EVERY TIME N IS CHANGED. RIGHT NOW N+2=21.                 *
C
C
C
      DIMENSION A(21,21)
      READ(5,10) N,D1,D2,D3
   10 FORMAT(I3,2X,3F4.4)
      CALL COMB(N,A)
      CALL POINTS(N,D1,A)
      CALL POINTS(N,D2,A)
      CALL POINTS(N,D3,A)
      STOP
      END
```

PROGRAM II

```
C*********************************************************
C                                                       *
CTHIS PROGRAM PLOTS THE FUNCTION FNX.THE INPUT*
C IS VALUES OF X AND THE CORRESPONDING VALUES *
C OF THE FUNCTION FNX.DATA FOR A PARTICULAR   *
C VALUE OF N AND THREE VALUES OF D IS GIVEN.  *
C THE THREE SETS OF DATA ARE SEPARATED BY A   *
C VALUE OF (0.0,0.0).SUBROUTINE SMOOTH IS USED*
C TO PLOT A SMOOTH CURVE FOR EACH SET OF DATA.*
C*********************************************************
C
C
         CALL PLOTS
         CALL PLOT(4.0,4.0,-3)
         CALL AXIS(0.0,0.0,'FNX',3,6.0,90.0,0.0,2.0)
         CALL AXIS(0.0,0.0,'X',-1,6.0,0.0,0.0,0.2)
C
C
C X AND Y AXES HAVE BEEN PLOTTED.SCALING USED IS:
C X AXIS, 1 UNIT=.2; Y AXIS, 1 UNIT=2.0.LENTH OF
C THE AXES IS 6 UNITS EACH.
C
C
  100    FORMAT(F12.8,5X,F12.8)
         DO 30 I=1,3
         READ(5,100) X1,Y1
         X1=X1*5
         Y1=Y1/2.0
         CALL SMOOTH(X1,Y1,0)
         READ(5,100) X1,Y1
   10    READ(5,100) X2,Y2
         IF ( (X2 .EQ. 0.0) .AND. (Y2 .EQ. 0.0) )
     *    GOTO 20
         X1=X1*5
         Y1=Y1/2.0
         CALL SMOOTH(X1,Y1,-2)
         X1=X2
         Y1=Y2
         GOTO 10
   20    X1=X1*5
         Y1=Y1/2.0
```

```
      CALL SMOOTH(X1,Y1,-25)
30    CONTINUE
      CALL PLOT(0.0,10.0,999)
      STOP
      END
```

APPENDIX II

x	$f_n(x)$
0.14999998	0.0
0.16999996	0.00016000
0.18999994	0.00128000
0.20999992	0.00431998
0.22999990	0.01023997
0.24999988	0.01999994
0.26999986	0.03455990
0.28999984	0.05487984
0.29999995	0.06749994
0.31999993	0.13025975
0.33999991	0.25877947
0.35999990	0.44441915
0.37999988	0.67853862
0.39999986	0.95249844
0.41999984	1.25765705
0.43999982	1.58537579
0.44999993	1.75499821
0.46999991	2.69915676
0.48999989	3.40907669
0.50999987	3.90491581
0.52999985	4.20683765
0.54999983	4.33499622
0.56999981	4.30955601
0.58999979	4.15067863
0.59999990	4.02749538
0.61999989	4.16465569
0.63999987	3.19318008
0.65999985	2.37961674
0.67999983	1.71053791
0.69999981	1.17250061
0.71999979	0.75206834
0.73999977	0.43578565

N+1 = 5

D = .15

POINTS OF THE PROBABILITY FUNCTION $f_n(x)$

x	$f_n(x)$
0.19999999	0.0
0.21999997	0.00016000
0.23999995	0.00128000
0.25999993	0.00431998
0.27999991	0.01023997
0.29999989	0.01999994
0.31999987	0.03455990
0.33999985	0.05487984
0.35999984	0.08191973
0.37999982	0.11663961
0.39999980	0.15999949
0.39999998	0.15999991
0.41999996	0.24015969
0.43999994	0.37887949
0.45999992	0.56751919
0.47999990	0.79743874
0.49999988	1.05999756
0.51999986	1.34655762
0.53999984	1.64847660
0.55999982	1.95711613
0.57999980	2.26383495
0.59999979	2.55999565
0.59999996	2.55999756
0.61999995	3.12975597
0.63999993	3.50847626
0.65999991	3.71631718
0.67999989	3.77343559
0.69999987	3.69999790
0.71999985	3.51615906
0.73999983	3.24207973
0.75999981	2.89792061
0.77999979	2.50384140
0.79999977	2.07999992
0.79999995	2.08000183
0.81999993	1.63296223
0.83999991	1.14687920
0.85999990	0.76831901
0.87999988	0.43384267
0.89999986	0.28000164
0.91999984	0.14336455
0.93999982	0.06048144

continued ...

x	$f_n(x)$
0.95999980	0.01792191
0.97999978	0.00224317
0.99999976	0.00001091

$$N+1 = 5$$

$$D = .2$$

x	$f_n(x)$
0.25000000	0.0
0.26999998	0.00016000
0.28999996	0.00128000
0.30999994	0.00431998
0.32999992	0.01023997
0.34999990	0.01999994
0.36999989	0.03455990
0.38999987	0.05487984
0.40999985	0.08191973
0.42999983	0.11663961
0.44999981	0.15999949
0.46999979	0.21295935
0.48999977	0.27647918
0.50000000	0.31250000
0.51999998	0.41605973
0.53999996	0.57097948
0.55999994	0.76861930
0.57999992	1.00033855
0.59999990	1.25749779
0.61999989	1.53145790
0.63999987	1.81357670
0.65999985	2.09521580
0.67999983	2.36773586
0.69999981	2.62249565
0.71999979	2.85085583
0.73999977	3.04417610
0.75000000	3.12500000
0.76999998	3.34435558
0.78999996	3.41587639
0.80999994	3.35971737
0.82999992	3.19603729
0.84999990	2.94499588
0.86999989	2.62675953
0.88999987	2.26147938
0.90999985	1.86931992
0.92999983	1.47044277
0.94999981	1.08500481
0.96999979	0.73316389
0.98999977	0.43508625

N+1 = 5

D = .25

x	$f_n(x)$
0.05000000	0.0
0.06999999	0.00000000
0.08999997	0.00000000
0.09999996	0.00000000
0.11999995	0.00000006
0.13999993	0.00000084
0.14999998	0.00000302
0.16999996	0.00002686
0.18999994	0.00021317
0.19999999	0.00056049
0.21999997	0.00308594
0.23999995	0.01525126
0.25000000	0.03141902
0.26999998	0.11402690
0.28999996	0.36735576
0.29999995	0.60745716
0.31999993	1.48858547
0.33999991	3.11749649
0.34999996	4.14516640
0.36999995	6.75558186
0.38999993	8.80857849
0.39999998	9.25242710
0.41999996	9.22718306
0.43999994	6.76822090
0.44999999	5.32125378
0.46999997	2.43540382
0.48999995	0.75956202

N+1 = 10

D = .05

POINTS OF THE PROBABILITY FUNCTION $f_n(x)$

x	$f_n(x)$
0.10000002	0.0
0.12000000	0.00000000
0.13999999	0.00000000
0.15999997	0.00000002
0.17999995	0.00000015
0.19999993	0.00000090
0.20000005	0.00000090
0.22000003	0.00000387
0.24000001	0.00001368
0.25999999	0.00004523
0.27999997	0.00014674
0.29999995	0.00044910
0.30000007	0.00044910
0.32000005	0.00124937
0.34000003	0.00317647
0.36000001	0.00762981
0.38000000	0.01752147
0.39999998	0.03807797
0.40000010	0.03807814
0.42000008	0.07751429
0.44000006	0.14859414
0.46000004	0.27097487
0.48000002	0.46976060
0.50000000	0.77021760
0.50000012	0.77021956
0.52000010	1.19187737
0.54000008	1.74831486
0.56000006	2.43155861
0.58000004	3.19505310
0.60000002	3.95596313
0.60000014	3.95596504
0.62000012	4.61495018
0.64000010	5.06993961
0.66000009	5.22136688
0.68000007	5.02256393
0.70000005	4.49936199
0.70000017	4.49935627
0.72000015	3.74134350
0.74000013	2.86602783
0.76000011	2.00880432
0.78000009	1.27870274

continued ...

x	$f_n(x)$
0.80000007	0.73132712
0.80000019	0.73134053
0.82000017	0.37004709
0.84000015	0.16275913
0.86000013	0.06070851
0.88000011	0.01852956
0.90000010	0.00433540
0.90000021	0.00435212
0.92000020	0.00074362
0.94000018	0.00006932
0.96000016	0.00004155
0.98000014	0.00002963

$N+1 = 10$

$D = .1$

x	$f_n(x)$
0.14999998	0.0
0.16999996	0.00000000
0.18999994	0.00000000
0.20999992	0.00000002
0.22999990	0.00000015
0.24999988	0.00000090
0.26999986	0.00000387
0.28999984	0.00001328
0.29999995	0.00002307
0.31999993	0.00006278
0.33999991	0.00015320
0.35999990	0.00034607
0.37999988	0.00074556
0.39999986	0.00155958
0.41999984	0.00318037
0.43999982	0.00629526
0.44999993	0.00874200
0.46999991	0.01637739
0.48999989	0.02949681
0.50999987	0.05122401
0.52999985	0.08613634
0.54999983	0.14060110
0.56999981	0.22284114
0.58999979	0.34261190
0.59999991	0.41985470
0.61999989	0.61553764
0.63999987	0.87394774
0.65999985	1.20217991
0.67999983	1.60206604
0.69999981	2.06673431
0.71999979	2.57788563
0.73999977	3.10473633
0.74999988	3.36125374
0.76999986	3.83015323
0.78999984	4.19707966
0.80999982	4.41191196
0.82999980	4.43454361
0.84999979	4.24462891
0.86999977	3.84883213
0.88999975	3.28310966
0.89999986	2.95496559

continued ...

x	$f_n(x)$
0.91999984	2.25514317
0.93999982	1.56464767
0.95999980	0.95775485
0.97999978	0.48789287
0.99999976	0.17744696

$N+1 = 10$

$D = .15$

x	$f_n(x)$
0.04700000	0.0
0.06699997	0.00000000
0.08699995	0.00000000
0.09399998	0.00000000
0.11399996	0.00000000
0.13399994	0.00000000
0.14099997	0.00000000
0.16099995	0.00000000
0.18099993	0.00000000
0.18799996	0.00000001
0.20799994	0.00000012
0.22799993	0.00000116
0.23499995	0.00000246
0.25499994	0.00001896
0.27499992	0.00012390
0.28199995	0.00023011
0.30199993	0.00122227
0.32199991	0.00558607
0.32899994	0.00918958
0.34899992	0.03464938
0.36899990	0.11329764
0.37599999	0.16598892
0.39599997	0.44957542
0.41599995	1.05748653
0.42299998	1.37989044
0.44299996	2.67675877
0.46299994	4.48677444
0.46999997	5.18875599
0.48999995	7.08026695
0.50999993	8.23807526
0.51699996	8.34980106
0.53699994	7.71578312
0.55699992	5.92393112
0.56399995	5.15394878
0.58399993	3.00089073
0.60399991	1.38145638
0.61099994	0.98988897
0.63099992	0.31209290
0.65099990	0.06820448
0.65799993	0.03635873
0.67799991	0.00498340

N+1 = 15

D = .047

POINTS OF THE PROBABILITY FUNCTION $f_n(x)$

x	$f_n(x)$
0.06699997	0.0
0.08699995	0.00000000
0.10699993	0.00000000
0.12699991	0.00000000
0.13399994	0.00000000
0.15399992	0.00000000
0.17399991	0.00000000
0.19399989	0.00000000
0.20099992	0.00000000
0.22099990	0.00000001
0.24099988	0.00000006
0.26099986	0.00000037
0.26799989	0.00000067
0.28799987	0.00000348
0.30799985	0.00001638
0.32799983	0.00006947
0.33499986	0.00011254
0.35499984	0.00042028
0.37499982	0.00143709
0.39499980	0.00450020
0.40199983	0.00657633
0.42199981	0.01838376
0.44199979	0.04732316
0.46199977	0.11221516
0.46899980	0.14892495
0.48899978	0.31654280
0.50899976	0.62017536
0.52899975	1.11945152
0.53599977	1.34994507
0.55599976	2.17830563
0.57599974	3.22858715
0.59599972	4.38687038
0.60299975	4.78222466
0.62299973	5.75334167
0.64299971	6.30015087
0.66299969	6.25321007
0.66999972	6.08820820
0.68999970	5.24847031
0.70999968	4.04568577
0.72999966	2.76617813
0.73699969	2.34974480

continued ...

x	$f_n(x)$
0.75699967	1.34495831
0.77699965	0.66483343
0.79699963	0.27903068
0.80399966	0.19713759
0.82399964	0.06425452
0.84399962	0.01611321
0.86399961	0.00323460
0.87099963	0.00134316
0.89099962	0.00036085
0.91099960	0.00007201
0.93099958	0.00073744
0.93799961	0.00046743
0.95799959	0.00026522
0.97799957	0.00075879
0.99799955	0.00034385

$N+1 = 15$

$D = .067$

x	$f_n(x)$
0.08700001	0.0
0.10699999	0.00000000
0.12699997	0.00000000
0.14699996	0.00000000
0.16699994	0.00000000
0.17400002	0.00000000
0.19400001	0.00000000
0.21399999	0.00000000
0.23399997	0.00000000
0.25399995	0.00000002
0.26100004	0.00000003
0.28100002	0.00000015
0.30100000	0.00000064
0.32099998	0.00000261
0.34099996	0.00000972
0.34800005	0.00001511
0.36800003	0.00005083
0.38800001	0.00016022
0.40799999	0.00047358
0.42799997	0.00131346
0.43500006	0.00184985
0.45500004	0.00472699
0.47500002	0.01139560
0.49500000	0.02593213
0.51499999	0.05573017
0.52200007	0.07187277
0.54200006	0.14311713
0.56200004	0.26937026
0.58200002	0.47906375
0.60200000	0.80468041
0.60900009	0.95179045
0.62900007	1.47828060
0.64900005	2.16368580
0.66900003	2.97978401
0.68900001	3.85405540
0.69600010	4.15341187
0.71600008	4.91685200
0.73600006	5.43187237
0.75600004	5.57915497
0.77600002	5.30327129
0.78300011	5.11030674

continued ...

x	$f_n(x)$
0.80300009	4.33485031
0.82300007	3.35004616
0.84300005	2.33553410
0.86300004	1.45004177
0.87000012	1.19043541
0.89000010	0.61420840
0.91000009	0.26759511
0.93000007	0.09427696
0.95000005	0.02484014
0.95700014	0.01443111
0.97700012	0.00188907
0.99700010	0.00010137

$N+1 = 15$

$D = .087$

x	$f_n(x)$
0.03000000	0.0
0.05000000	0.00000000
0.06000000	0.00000000
0.07999998	0.00000000
0.08999997	0.00000000
0.10999995	0.00000000
0.12000000	0.00000000
0.13999999	0.00000000
0.14999998	0.00000000
0.16999996	0.00000000
0.18000001	0.00000000
0.19999999	0.00000000
0.20999998	0.00000001
0.22999996	0.00000031
0.24000001	0.00000134
0.25999999	0.00002022
0.26999998	0.00007120
0.28999996	0.00072930
0.30000001	0.00212499
0.31999999	0.01499309
0.32999998	0.03632654
0.34999996	0.17751634
0.36000001	0.35787213
0.38000000	1.20739174
0.38999999	2.01778603
0.40999997	4.64309311
0.42000002	6.37708950
0.44000000	9.77926826
0.44999999	10.87393570
0.46999997	10.69343281
0.48000002	9.39288330
0.50000000	5.56202221
0.50999999	3.70538807
0.52999997	1.17961025
0.54000002	0.55092895
0.56000000	0.07719499
0.56999999	0.01934515
0.58999997	0.02313974

continued ...

x	$f_n(x)$
0.73999995	1.39158344
0.75000000	0.93759954
0.76999998	0.36744744
0.78999996	0.11698687
0.80000001	0.05807549
0.81999999	0.01151742
0.83999997	0.00000341
0.84999996	0.00062970
0.86999995	0.00609926
0.88999993	0.00370306
0.89999998	0.00293845
0.91999996	0.00327434
0.93999994	0.00136012
0.94999999	0.00408846
0.96999997	0.00822474
0.98999995	0.00132161

$N+1 = 20$

$D = .03$

POINTS OF THE PROBABILITY FUNCTION $f_n(x)$

x	$f_n(x)$
0.05000000	0.0
0.06999999	0.00000000
0.08999997	0.00000000
0.09999996	0.00000000
0.11999995	0.00000000
0.13999993	0.00000000
0.14999998	0.00000000
0.16999996	0.00000000
0.18999994	0.00000000
0.19999999	0.00000000
0.21999997	0.00000000
0.23999995	0.00000000
0.25000000	0.00000000
0.26999998	0.00000001
0.28999996	0.00000007
0.29999995	0.00000019
0.31999993	0.00000131
0.33999991	0.00000819
0.34999996	0.00001954
0.36999995	0.00010148
0.38999993	0.00046891
0.39999998	0.00096560
0.41999996	0.00376274
0.43999994	0.01311393
0.44999999	0.02350425
0.46999997	0.06948674
0.48999995	0.18411052
0.50000000	0.28759397
0.51999998	0.64599752
0.53999996	1.29794502
0.54999995	1.76352024
0.56999993	2.98703003
0.58999991	4.50006390
0.59999996	5.28046227
0.61999995	6.62814331
0.63999993	7.32459641
0.64999998	7.32770157
0.66999996	6.61493206
0.68999994	5.16995997
0.69999999	4.31777668
0.71999997	2.67052364

N+1 = 20

D = .05

x	$f_n(x)$
0.06999999	0.0
0.08999997	0.00000000
0.10999995	0.00000000
0.12999994	0.00000000
0.13999999	0.00000000
0.15999997	0.00000000
0.17999995	0.00000000
0.19999993	0.00000000
0.20999998	0.00000000
0.22999996	0.00000000
0.24999994	0.00000000
0.26999992	0.00000000
0.27999997	0.00000000
0.29999995	0.00000000
0.31999993	0.00000002
0.33999991	0.00000009
0.34999996	0.00000019
0.36999995	0.00000093
0.38999993	0.00000418
0.40999991	0.00001658
0.41999996	0.00003246
0.43999994	0.00011759
0.45999992	0.00039573
0.47999990	0.00123911
0.48999995	0.00213511
0.50999993	0.00601482
0.52999991	0.01580729
0.54999989	0.03877158
0.55999994	0.05917258
0.57999992	0.13087237
0.59999990	0.27005053
0.61999989	0.51951927
0.62999994	0.70165992
0.64999992	1.21254730
0.66999990	1.94695377
0.68999988	2.89853191
0.69999993	3.43502045
0.71999991	4.54229927
0.73999989	5.52599525
0.75999987	6.15758801
0.76999992	6.27902699

continued ...

x	$f_n(x)$
0.78999990	6.06845284
0.80999988	5.28544044
0.82999986	4.10895729
0.83999991	3.45872498
0.85999990	2.21260929
0.87999988	1.21485519
0.89999986	0.55688423
0.90999991	0.34862763
0.92999989	0.11256355
0.94999987	0.02570609
0.96999985	0.00372719
0.97999990	0.00123571
0.99999988	0.00010629

$N+1 = 20$

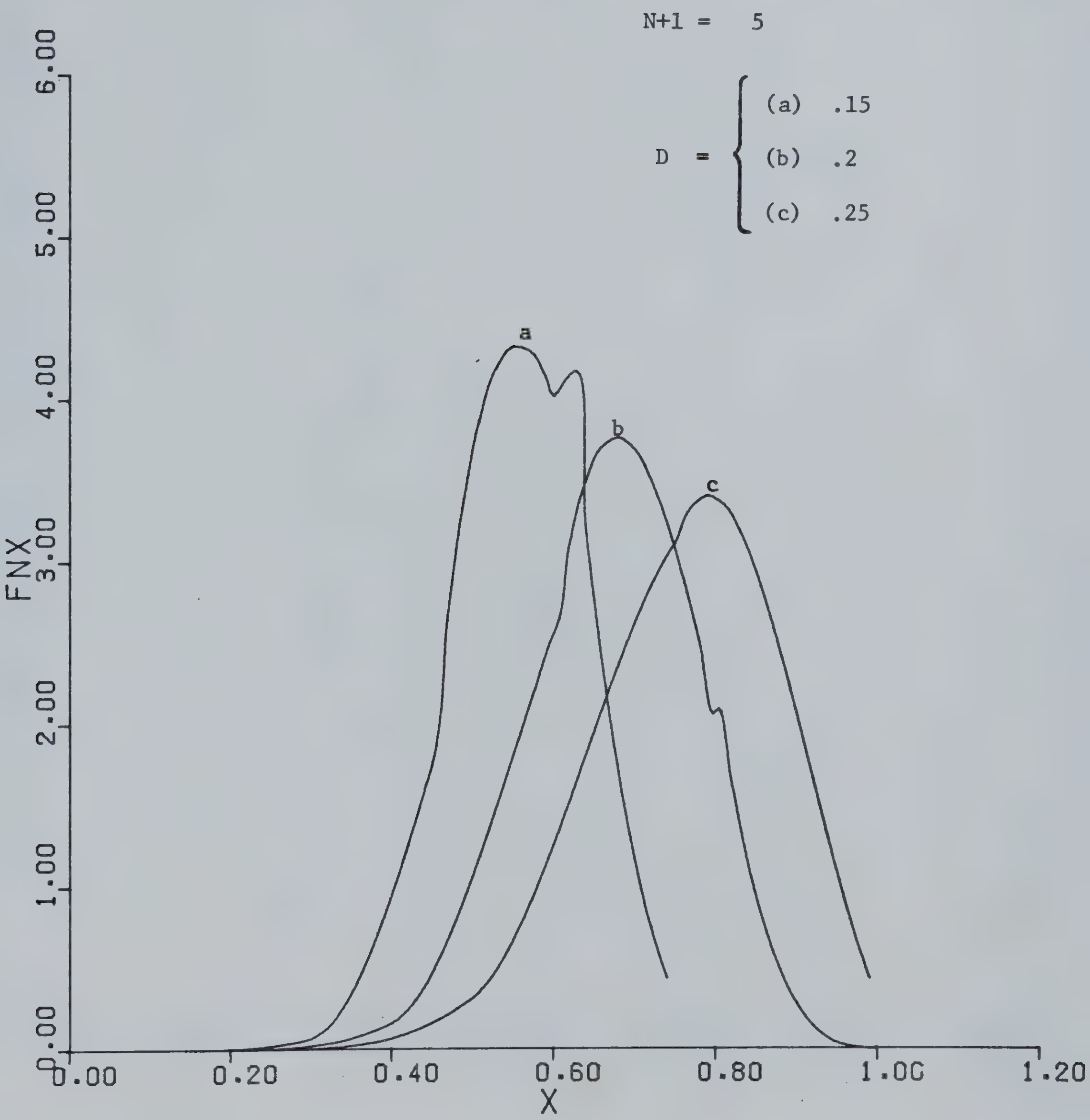
$D = .07$

APPENDIX III

PROBABILITY FUNCTION $f_n(x)$ OF THE MEASURE

OF A RANDOM CIRCULAR SET

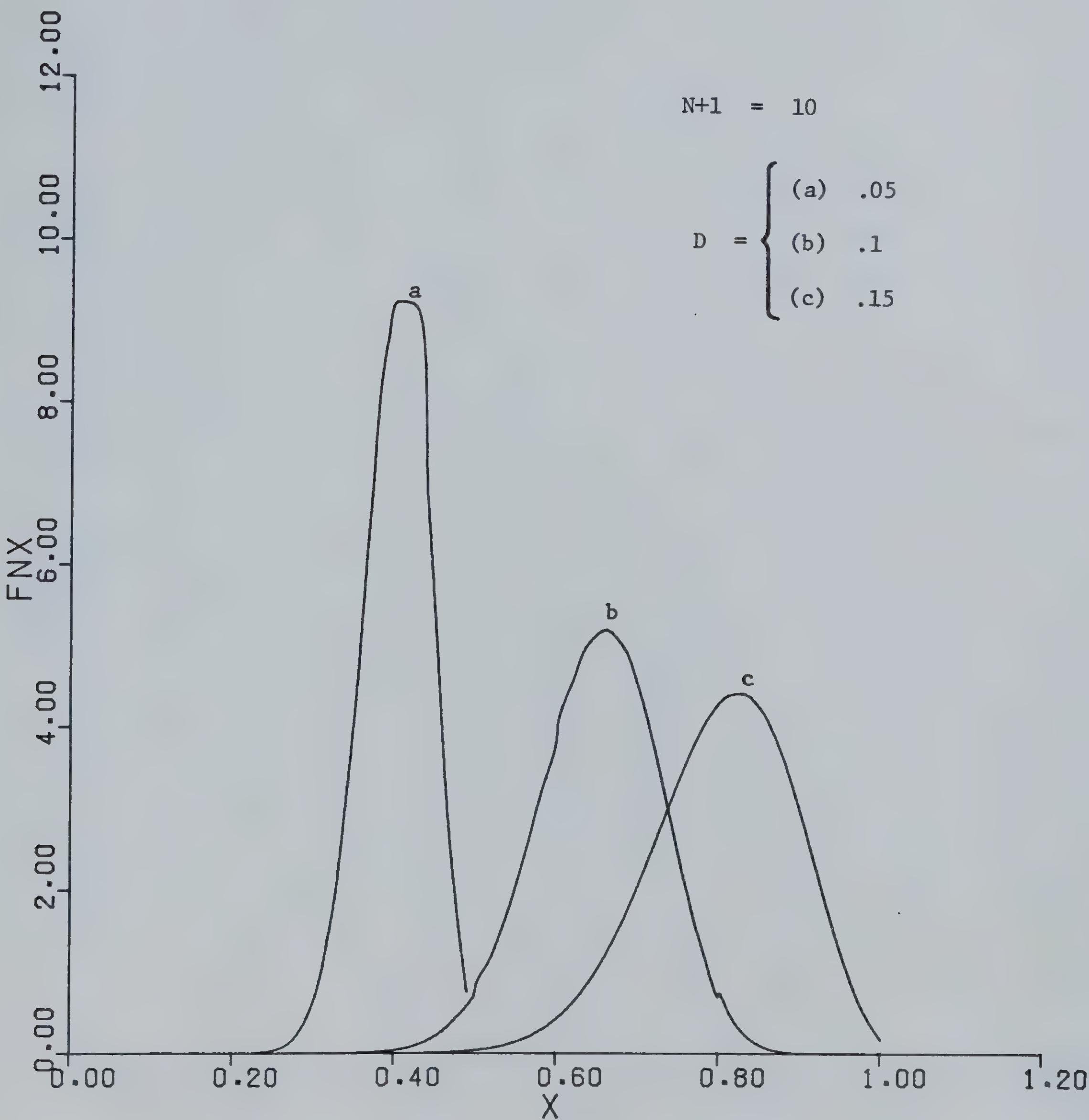
PLOT 1



PROBABILITY FUNCTION $f_n(x)$ OF THE MEASURE

OF A RANDOM CIRCULAR SET

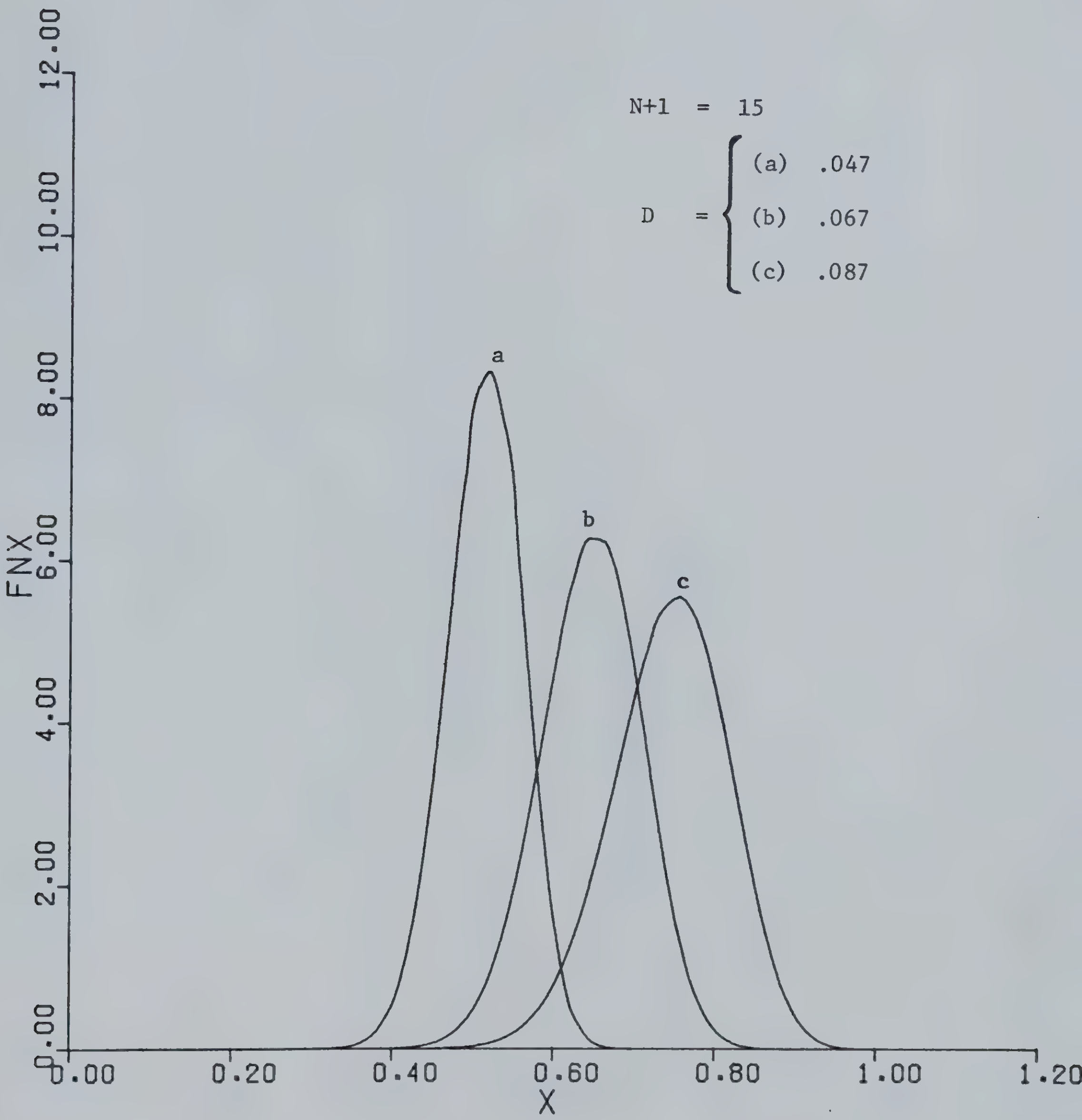
PLOT 2



PROBABILITY FUNCTION $f_n(x)$ OF THE MEASURE

OF A RANDOM CIRCULAR SET

PLOT 3



PROBABILITY FUNCTION $f_n(x)$ OF THE MEASURE

OF A RANDOM CIRCULAR SET

PLOT 4